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A THEORY OF SYSTEMS WITH UNILATERAL CONSTRAINTS†

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The realization of a unilateral constraint is considered in a situation in which the stiffness and coefficient of viscosity and the added masses tend to infinity simultaneously in a consistent manner. The main result is that limiting motions exist, which are identical on the boundary with the motions of a holonomic system with fewer degrees of freedom. However, a special effect, not present in the classical model, occurs here, namely, a delay in the time at which the constraint is released.

The formal-axiomatic approach to the rigorous mathematical theory of the dynamics of systems with constraints has an obvious drawback: the origin and physical meaning of the basic principles remain unclear, as do the limits of applicability of the theoretical models. In that respect, one should prefer the constructive method outlined by Klein, Prandtl and Lecornieux in connection with Painlevé's paradoxes of dry friction [1]: instead of a holonomic constraint, one considers a field of elastic forces directed toward the appropriate surface. It turns out that as the stiffness tends to infinity the motions of the "free" system tend to motions of a system with a holonomic constraint. Rigorous formulations were given by Courant; the proof for a conservative force field may be found in [2] and a general proof in [3].

The problem of realizing holonomic constraints by elastic forces and forces of viscous friction was considered in [4]. The stabilization of numerical methods for integrating the equations of motion with bilateral constraints, using additional conservative and dissipative forces, was studied in [5].

An analogous approach was developed for unilateral constraints in [6] (see also [7]): a non-retentive holonomic constraint is replaced by a field of elastic forces directed towards the boundary, after which the stiffness is allowed to go to infinity. It turns out that trajectories that cut the boundary transversely tend to trajectories of a limiting system with impacts. This result has proved effective in investigating the stability of periodic trajectories and the evolution of vibro-impact systems [6–8]. The case in which the velocity of the system at the starting time touches the boundary surface was considered in [6, 9]. It turns out that if the stiffness is increased without limit, the motion of the "free" system tends to motion along the boundary, provided that allowance is made for the possibility of releasing the constraint. The case in which the applied forces are conservative was considered in [6], and the limit theorem was proved in [9] without assuming that the generalized forces are conservative.

In this paper a more general situation will be considered: the half-space is replaced by a Kelvin–Voigt medium and the stiffness and elasticity of the "free" system will be sent to infinity in a consistent manner.

1. LIMIT THEOREM

Let $x_1, \ldots, x_n$ be generalized coordinates of a mechanical system, let

$$T = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x) x_i^{\cdot} x_j^{\cdot} \tag{1.1}$$

be the kinetic energy and let $F_1, \ldots, F_n$ be generalized forces, which depend on x and $x^{\cdot}$. We introduce a unilateral constraint, given by the inequality

$$f(x) \geqslant 0 \tag{1.2}$$

where f is a smooth function such that $df \neq 0$ at points where $f = 0$. We put $\Sigma = \{x: f(x) = 0\}$. Clearly,

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Σ is a regular hypersurface. Let us consider a motion $x(t)$ of the system with initial data

$$x(0)=x_0,\quad x^{\cdot}(0)=\upsilon,\quad f(x_0)=0,\quad \frac{\partial f}{\partial x}(x_0)\upsilon=0 \tag{1.3}$$

The last condition means that the velocity vector υ touches Σ. Otherwise this will be a motion with impacts.

In motion on the surface Σ the function $x(t)$ will satisfy the Lagrange equations with multiplier

$$\left(\frac{\partial T}{\partial x^{\cdot}}\right)^{\cdot}-\frac{\partial T}{\partial x}=F+R,\quad R=\lambda\frac{\partial f}{\partial x},\quad f=0 \tag{1.4}$$

The force R is a reaction, which is a covector, vanishing at vectors (1.3) tangent to Σ: $R\upsilon = 0$. With the covector $R = \{R_i\}$ we associate the vector $r = \{r^i\}$ with components

$$r^i=\sum_{j=1}^{n} a^{ij}R_j$$

where $\| a^{ij} \|$ is the matrix inverse to $\| a_{ij} \|$. The vector r is clearly orthogonal to Σ in the intrinsic metric defined by the kinetic energy (1.1). Let μ be the algebraic value to the projection of r onto the normal to Σ; $| \mu |$ equals the magnitude of the reaction.

For the motion under consideration, $t \to x(t) \in \Sigma$, the Lagrange multiplier λ and the projection μ are certain continuous functions of time. If $\mu(t) \geqslant 0$ for $0 \leqslant t \leqslant \tau$, then, considered from the standpoint of the classical model of dynamics with a unilateral constraint in the interval $[0, \tau]$, the system will move along the surface Σ. But if $\mu(t) < 0$ in some interval $t \in (\tau, \tau + \delta)$, $\delta > 0$, then when $t = \tau$ the system will leave Σ and become free.

Following [6], let us replace the unilateral constraint (1.2) by a viscoelastic Kelvin–Voigt medium filling the half-space $f(x) \leqslant 0$. The elastic properties of the medium are defined by its potential energy

$$V_N=\gamma N f^2/2 \tag{1.5}$$

and its viscosity by the Rayleigh dissipative function

$$\Phi_N=\frac{\beta N}{2}\left(\frac{\partial f}{\partial x}x^{\cdot}\right)^2 \tag{1.6}$$

where β and γ are non-negative constants. In the domain $f(x) \geqslant 0$ the functions V_N and Φ_N are assumed to vanish. The potential (1.5) is usually used in the problem of realizing holonomic constraints (cf. [2, 3]). The dissipative forces $-\partial\Phi_N/\partial x^{\cdot}$, corresponding to the Rayleigh function (1.6), perform no work when the system is moving along the surface f = const.

When the medium is deformed, its particles are displaced in a direction transverse to Σ. Therefore, to describe the dynamics of the system in the half-space $f(x) \leqslant 0$, the effect of the added masses must be taken into account. This effect will be modelled by a variation in the inertial properties of the system: instead of the kinetic energy (1.1), we take the energy

$$T_N=T+\frac{\alpha N}{2}\left(\frac{\partial f}{\partial x}x^{\cdot}\right)^2,\quad \alpha=\text{const}>0 \tag{1.7}$$

Clearly, at all values of $N \geqslant 0$ the quadratic form (1.7) is positive-definite. In the domain $f \geqslant 0$ we must, of course, put $T_N \equiv T$. In what follows the parameter α will tend to zero. Regardless of physical feasibility, the introduction of added masses is related to the regulation of the passage to the limit $N \to \infty$.

Thus, the motion of the "free" system is described by the equations

$$\left(\frac{\partial T_N}{\partial x^{\cdot}}\right)^{\cdot}-\frac{\partial T_N}{\partial x}=F-\frac{\partial V_N}{\partial x}-\frac{\partial \Phi_N}{\partial x^{\cdot}} \tag{1.8}$$

In the domain $f \geqslant 0$ these equations are identical with the usual Lagrange equations

$$\left(\frac{\partial T}{\partial x^{\cdot}}\right)^{\cdot} - \frac{\partial T}{\partial x} = F \tag{1.9}$$

Remark. Equations (1.8) are formulated on the assumption that the system is "embedded" in a Kelvin–Voight medium filling the half-space $f(x) < 0$. One can consider a different model of the interaction of the system with a viscoelastic barrier: the barrier may exert a reactive force $-\partial V_N/\partial x - \partial \Phi_N/\partial x^{\cdot}$ only in the direction of increasing f. At other times it is assumed to vanish. This model was developed for impact theory in [7]. An example of its use to determine when the system is released from the constraint will be considered in Section 6.

Let $x_N(t)$ be the solution of Eqs (1.8)–(1.9) with initial data $x_N(0) = x_0, x_N^{\cdot}(0) = \upsilon$ which satisfies conditions (1.3). Let us consider the auxiliary second-order equation

$$\alpha z^{\cdot\cdot} + \beta z^{\cdot} + \gamma z = -\mu(t) \tag{1.10}$$

where $\mu(t)$ is, as defined above, the algebraic value of the magnitude of the reaction of the constraint when the system is moving over the surface Σ with the same initial data. Let $z(t)$ be the solution of Eq. (1.10) defined by the conditions $z(0) = 0, z^{\cdot}(0) = 0$.

Theorem 1. Assume that Eqs (1.4) have a solution $x(t), x(0) = x_0, x^{\cdot}(0) = \upsilon$, in the interval $0 \leqslant t \leqslant \tau + \delta, \delta > 0$, and $z(t) < 0$ for $0 < t < \tau$. Then for all $0 \leqslant t \leqslant \tau$ the limit

$$x^{\wedge}(t) = \lim_{N \to \infty} x_N(t) \tag{1.11}$$

exists and the function $x^{\wedge}(t)$ satisfies Eq. (1.4). If in addition τ is the first simple zero if $z(t)$ and $\mu(\tau) < 0$, then the limit (1.11) exists in some larger interval $0 \leqslant t \leqslant \tau + \delta_1, \delta_1 > 0$, where $x^{\wedge}(t)$ satisfies system (1.9) for $\tau < t \leqslant \tau + \delta_1$ and $f(x^{\wedge}(t)) > 0$.

It should be emphasized that the case $\tau = 0$ is not excluded. The function $t \to x^{\wedge}(t)$ may be regarded as the motions of a mechanical system with kinetic energy T and a unilateral constraint, driven by given generalized forces F, in the limit model of the motion. Since this model depends on the parameters α, β and γ, it may be called the (α, β, γ)-model. Seen in the context of all these models, the motion of the system on the surface Σ obeys the same law. They differ only in the conditions for release of the constraint. If one multiplies the parameters α, β and γ by any positive number, the resulting model is clearly the same.

2. PROOF OF THE MAIN THEOREM

In the neighbourhood of Σ we introduce semi-geodesic coordinates $x_1, \ldots, x_n$, in which $f \equiv x_n$ and the metric (1.1) becomes

$$T = T^* + \frac{1}{2} a_{nn} x_n^{\cdot 2}, \quad T^* = \frac{1}{2} \sum_{i,j=1}^{n-1} a_{ij}(x) x_i^{\cdot} x_j^{\cdot}$$

We know (see, e.g. [7, 10]) that such coordinates always exist. Equations (1.4) are written in terms of them as

$$\left(\frac{\partial T^*}{\partial x_i^{\cdot}}\right)^{\cdot} - \frac{\partial T^*}{\partial x_i} - \frac{1}{2} \frac{\partial a_{nn}}{\partial x_i} x_n^{\cdot 2} = F_i, \quad i < n$$

$$(a_{nn} x_n^{\cdot})^{\cdot} - \frac{\partial T^*}{\partial x_n} - \frac{1}{2} \frac{\partial a_{nn}}{\partial x_n} x_n^{\cdot 2} = F_n + R, \quad x_n = 0 \tag{2.1}$$

Suppose that the motion of the system in the interval $0 \leqslant t \leqslant \tau$ is confined to the surface Σ. Then it follows from the last equation of system (2.1) that

$$R = -\left(F_n + \frac{\partial T^*}{\partial x_n} \right)_0 \tag{2.2}$$

The zero subscript means that the expression is considered at the point $x_n = 0, x_n^{\cdot} = 0$.

It is clear that (2.2) is precisely μ. At the same time, the first equation of (2.1) takes the form of the ordinary Lagrange equations for a system with $n - 1$ degrees of freedom

$$\left(\frac{\partial T_0^*}{\partial x_i^{\cdot}} \right)^{\cdot} - \frac{\partial T_0^*}{\partial x_i} = (F_i)_0, \quad i < n \tag{2.3}$$

Let us write Eqs (1.8) taking into account the fact that now $f = x_n$. The first group of equations of (2.1) remains unchanged, while the second becomes a little more complicated

$$[(\alpha N + a_{nn}) x_n^{\cdot}]^{\cdot} - \frac{\partial T^*}{\partial x_n} - \frac{1}{2} \frac{\partial a_{nn}}{\partial x_n} x_n^{\cdot 2} = F_n - \beta N x_n^{\cdot} - \gamma N x_n \tag{2.4}$$

It should be borne in mind that this equation only holds when $x_n < 0$. We put $\varepsilon = 1/N$ and divide both sides of (2.4) by N

$$\alpha x_n^{\cdot\cdot} + \beta x_n^{\cdot} + \gamma x_n = \varepsilon \left[F_n + \frac{\partial T}{\partial x_n} + \frac{1}{2} \frac{\partial a_{nn}}{\partial x_n} x_n^{\cdot 2} - (a_{nn} x_n^{\cdot})^{\cdot} \right] \tag{2.5}$$

We solve the first group of equations of (2.1) and (2.5) simultaneously. When $t = 0$ we have: $x_i(0) = x_i^0, x_i^{\cdot}(0) = \upsilon_i \ (i < n) x_n(0), x_n^{\cdot}(0) = 0$. Since the right-hand sides of the system are analytic functions of ε, we can use small parameters for the solution

$$x_i(t, \varepsilon) = x_i^0(t) + \varepsilon x_i^1(t) + \ldots, \quad i < n$$
$$x_n(t, \varepsilon) = \varepsilon x_n^1(t) + \ldots \tag{2.6}$$

The functions $x_i^0(t)$, $i < n$ satisfy system (2.3) with initial data $x_i(0) = x_i^0, x_i^{\cdot}(0) = \upsilon_i$, while the other functions $x_i^1, x_i^2, \ldots$ vanish at $t = 0$ together with their derivatives. Note that, by (2.2), the bracketed expression on the right of (2.5) is identical with $-R(t)$ at $\varepsilon = 0$. Consequently, $x_n^1(t)$ is a solution of Eq. (1.1) with zero initial data. By assumption, $x_n^1(t) < 0$ in the interval $0 < t < \tau$. Consequently, for small $\varepsilon > 0$, the coordinate x_n is negative if $0 < t < \tau$. In this time interval, therefore, the expansions (2.6) are indeed valid. Letting $\varepsilon \to 0$, we obtain the first part of Theorem 1.

If τ is the first simple zero of the function $z(t)$ $(z^{\cdot}(\tau) > 0)$, then $x_n(t, \varepsilon)$ has a zero $\tau_\varepsilon = \tau + O(\varepsilon)$ at small values of ε. Since $\mu(\tau) < 0$ by assumption, it follows by continuity that the function μ is negative in some small neighbourhood of τ. Thus, at $t = \tau_\varepsilon$

$$x_n = 0, \ x_n^{\cdot} = O(\varepsilon)$$

In addition, $\mu(\tau_\varepsilon) < 0$. Consequently, for small $\varepsilon > 0$ and $t > \tau_\varepsilon$ the system will move in the half-space $x_n > 0$. Letting ε tend to zero and remembering that the solutions are continuous with respect to the initial data, we obtain the desired conclusion.

3. DELAY OF THE RELEASE TIME OF THE CONSTRAINT

If $\alpha = \beta = 0$, then, as shown in [9], the limiting function $x^{\wedge}(t)$ exists and is a motion of the system from the classical standpoint. In particular, the first zero τ of $\mu(t)$, when $\mu^{\cdot}(\tau) < 0$, is the time of release of the constraint. By Theorem 1, in the (α, β, γ)-model the system certainly moves on the surface Σ until the solution of Eq. (1.10) with zero initial data becomes negative.

Proposition 1. If $\beta^2 \geqslant 4\alpha\gamma$ and $\mu(t) > 0$ for $0 < t < \tau$, then $z(t) < 0$ for all $0 < t < \tau + \kappa$, where $\kappa > 0$.

Proof. If $\beta^2 \geqslant 4\alpha\gamma$, the roots of the characteristic equation $\alpha\lambda^2 + \beta\lambda + \gamma = 0$ are real. Denote them by λ_1 and λ_2.

Let $\lambda_1 > \lambda_2$ (the treatment of the case $\lambda_1 = \lambda_2$ is analogous). Then the solution of Eq. (1.10) with zero initial data is

$$z(t) = -\frac{e^{\lambda_1 t}}{\alpha(\lambda_1-\lambda_2)}\int_0^t e^{-\lambda_1 s}\mu(s)ds + \frac{e^{\lambda_2 t}}{\alpha(\lambda_1-\lambda_2)}\int_0^t e^{-\lambda_2 s}\mu(s)ds$$

For small $t > 0$, obviously, $z(t) < 0$. Let $t = \xi$ be the first positive zero of $z(t)$. Then

$$\int_0^\xi e^{-\lambda_1 s}\mu(s)ds\left[\int_0^\xi e^{-\lambda_2 s}\mu(s)ds\right]^{-1} = \frac{e^{-\lambda_1\xi}}{e^{-\lambda_2\xi}}$$

Let $\xi \leqslant \tau$. Then, by assumption, $\mu > 0$ and so, by Cauchy's mean-value theorem, the interval $(0, \xi)$ contains a point η such that $e^{(\lambda_1-\lambda_2)\eta} = e^{(\lambda_1-\lambda_2)\xi}$. Since $\lambda_1 > \lambda_2$ and $\eta < \xi$, this is a contradiction.

Corollary. If $\beta > 0$ and α is small, one has the phenomenon of persistence of the constraint.

Proposition 2. Let $\beta^2 < 4\alpha\gamma$ and $\omega^2 = 4\alpha\gamma - \beta^2$, $\omega > 0$. If $\mu(t) > 0$ for $0 < t < \tau \leqslant \pi/\omega$, then $z(t) < 0$ for all $0 < t < \tau + \kappa$, $\kappa > 0$.

This is proved in the same way as Proposition 1.

One should not think that $z(t)$ always has a zero to the right of the first zero of $\mu(t)$. Here is a simple counter-example: $z^{\cdot\cdot} + z = -\mu(t)$, where $\mu(t) = 1$, $0 \leqslant t \leqslant 3\pi/2$, $\mu(t) = 1/2$, $t > 3\pi/2$.

The solution of this equation with zero initial data, for $t > 3\pi/2$, has the form $z(t) = (-1 + \sin t)/2 + \cos t$, i.e. $z(2\pi) = 1/2$.

4. REALIZATION OF A CONSTRAINT BY ELASTIC FORCES

We will consider the important special case in which $\beta = 0$, $\alpha = \nu^2$, $\nu \to 0$. We may assume without loss of generality that $\gamma = 1$. It has been shown [9] that, in the limiting case $\alpha = 0$, the function $x^{\wedge}(t)$ describes the motion of a system with a unilateral constraint in the classical model.

For small ν Proposition 2 does not provide significant information about the properties of the solution of the equation

$$\nu^2 z^{\cdot\cdot} + z = -\mu(t) \tag{4.1}$$

with zero initial data. This solution has the form

$$z(t,\nu) = -\sin\frac{t}{\nu}\int_0^t \frac{\mu(\tau)}{\nu}\cos\frac{\tau}{\nu}d\tau + \cos\frac{t}{\nu}\int_0^t\frac{\mu(\tau)}{\nu}\sin\frac{\tau}{\nu}d\tau$$

Integration by parts yields the following asymptotic representation

$$z(t,\nu) = \mu(0)\cos\frac{t}{\nu} - \mu(t) + O(\nu) \tag{4.2}$$

If $\mu(0) \neq 0$, this function will have no limit as $\nu \to 0$, because it oscillates rapidly. However

$$\lim_{t_1,t_2\to t}\lim_{\nu\to 0}\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} z(\tau,\nu)d\tau = -\mu(t) \tag{4.3}$$

Thus, after averaging, we obtain $-\mu(t)$ in the limit. The same value is obtained from (4.1) by formally substituting $\nu = 0$.

Let us proceed now to Courant's problem of the realization of a bilateral constraint $f(x) = 0$, assuming that, apart from elastic forces with potential energy $Nf^2/2$, there are added masses, which increase the kinetic energy by $\nu^2N(f^{\cdot})^2/2$, where ν is small. We must put $\gamma = 1$, $\beta = 0$, $\alpha = \nu^2$ in Eq. (2.4). On the right we obtain an additional term $-Nx_n$. As $N \to \infty$ we obtain an indeterminate expression of the type "$\infty \cdot 0$". It follows from the results of Section 2 that this term tends to the function $z(t, \nu)$ as $N \to \infty$. Letting $\nu \to \infty$ and regularizing (4.3), we see that the term $-Nx_n$ tends to the reaction of the holonomic constraint $f = x_n = 0$. This result was obtained differently in [3].

If $\mu(0) > 0$, then, for small $t > 0$, we have $f(x_N(t)) < 0$: the system is in a "forbidden" domain and is subject to large elastic forces that tend to push it out into the half-space $f \geqslant 0$. One may ask, how long can the system stay in the domain $f(x) < 0$? The asymptotic formula (4.2) gives some non-trivial estimates: if $\mu(t) > \mu(0)$, then for small $\varepsilon = 1/N$ and ν the trajectory $t \to x_N(t)$ of the system is in the domain $f(x) < 0$. Indeed, by (2.6), $x_n(t) = \varepsilon z(t, \nu) + o(\varepsilon)$. It remains to observe that if $\mu(t) > \mu(0)$, then, for small ν, the function $z(t, \nu)$ will be negative.

5. REALIZATION OF A CONSTRAINT BY ANISOTROPIC FRICTION

We will consider one more important special case: $\gamma = 0$, $\alpha \to 0$. We may assume without loss of generality that $\beta = 1$. Under these assumptions the solution of Eq. (1.10) with initial data zero is found from the relation

$$\dot z = -\frac{e^{-t/\alpha}}{\alpha}\int_0^t \mu(s)e^{s/\alpha}ds \tag{5.1}$$

Let us investigate the behaviour of the function $z(t, \alpha)$ as $\alpha \to 0$.

Proposition 3. If μ is a smooth function (say, C^2), then for $t \in (0, \tau]$

$$\lim_{\alpha\to 0} \dot z(t,\alpha) = -\mu(t) \tag{5.2}$$

$$\lim_{\alpha\to 0} z(t,\alpha) = -\int_0^t \mu(s)ds \tag{5.3}$$

Indeed, integrating (5.1) twice by parts, we obtain the formula

$$\dot z = -\mu(t) + e^{-t/\alpha}\mu(0) + \alpha\left[e^{-t/\alpha}\dot\mu(s)e^{s/\alpha}\Big|_0^t - e^{-t/\alpha}\int_0^t \ddot\mu\, e^{s/\alpha}ds\right] \tag{5.4}$$

To estimate the integral in this formula we use Bonnet's form of the mean-value theorem

$$\int_0^t \ddot f e^{s/\alpha}ds = \int_0^\xi \ddot f ds + e^{t/\alpha}\int_\xi^t \ddot f ds, \quad \xi \in [0,t]$$

Since $\mu \in C^2$, the function

$$e^{-t/\alpha}\int_0^t \ddot\mu\, e^{s/\alpha}ds$$

is bounded over any finite time interval. It remains to let $\alpha \to 0$ in (5.4).

If $\mu(0) \neq 0$, the convergence in the formula is not uniform: $\dot z \to 0$ at $t = 0$. In the general case, therefore, the truth of (5.3) does not follow from (5.2). However

$$\int_0^t e^{-s/\alpha}ds = \alpha(1 - e^{-t/\alpha})$$

and the bracketed expression in (5.4) is uniformly bounded. Hence the integral of the right-hand side tends to the integral of $-\mu$ as $\alpha \to 0$, which it was required to prove.

Let us assume that $\mu(t) > 0$ for small $t > 0$. Then, obviously, $f(x_N(t)) < 0$. Since $x_n(t, \varepsilon) = \varepsilon z(t) + o(\varepsilon)$, it follows that for small ε we have $x_n < 0$ if $z(t) < 0$. Let τ be the first zero of the function (5.3). Clearly, $\mu(\tau) \leqslant 0$. Most typically, $\mu(\tau) < 0$. Then τ is a simple zero of $z(t)$. Consequently, in the limit when $\varepsilon = 1/N \to 0$, the coordinate x_n vanishes for the first time, and moreover $\mu(\tau) < 0$. Therefore, for $t > \tau$ the system will leave the surface Σ and its dynamics will be described by Eqs (1.9).

We finally arrive at the following model of the motion. If $\mu(0) > 0$, the system begins to move over Σ until the mean value of the reaction of the constraint vanishes for the first time. If the reaction is negative at that instant (this is the typical case), the system becomes free. If the trajectory of the "free" system then cuts the surface Σ transversely, there will be an absolutely inelastic impact: the normal component of the velocity will vanish [6].

6. LIMIT THEOREM IN THE CASE OF ANISOTROPIC FRICTION

Let us consider the case in which $\alpha = 0$ and $\gamma = 0$ (Eqs (1.8) involve only additional forces of viscous friction). It turns out that as $N \to \infty$ the solutions of Eqs (1.8) tend to the motion of the system described in Section 5. Let $x_N(t)$ be the solution of Eqs (1.8)–(1.9) with the initial data $x_N(0)$, $x_N^{\cdot}(0)$ that satisfy (1.3), and let τ be the first simple zero of the function (5.3).

Theorem 2. A $\delta > 0$ exists such that, in the interval $0 \leqslant t \leqslant \tau + \delta$, the limit

$$x_*(t) = \lim_{N\to\infty} x_N(t) \tag{6.1}$$

exists, and moreover, for $0 \leqslant t \leqslant \tau$ the function $x_*(t)$ satisfies Eqs (1.4), but for $t > \tau + \delta$ it satisfies Eqs (1.9) and the inequality $f(x_*(t)) > 0$.

Proof. In the neighbourhood of Σ, we introduce semi-geodesic coordinates $x_1, \ldots, x_n$ (as in Section 2). We first consider a simpler case: the realization of a bilateral constraint $f(x) = 0$ by viscous friction forces. Let $x_N(t)$ be the solution of Eqs (1.8) (in which $\alpha = \gamma = 0$, $\beta = 1$) with initial data (1.3).

Over an interval of time, the solutions $x_N(t)$ of the singular equations (1.8) will have a limit $x^\wedge(t)$ [11, 12], and this limit function will be a solution of system (1.4). Since the initial data satisfy (1.3), the following asymptotic formulae hold (see, e.g. [13])

$$\begin{aligned}(x_N(t))_k &= x_k^\wedge(t) + O(\varepsilon) \quad (k < n)\\ (x_N(t))_n &= \varepsilon x_n^1(t) + o(\varepsilon), \quad \varepsilon = 1/N\end{aligned} \tag{6.2}$$

Substituting (6.2) for x_n into (2.5), we obtain

$$\varepsilon(x_n^1)^{\cdot} = \varepsilon(F_n + \partial T/\partial x_n)x^\wedge(t) + o(\varepsilon)$$

Therefore

$$x_n^1 = -\int_0^t \mu(s)ds \tag{6.3}$$

By assumption, if $0 < t < \tau_\varepsilon$ ($\tau_\varepsilon = \tau + O(\varepsilon)$) and ε is small, the coordinate x_n is negative and vanishes at time τ_ε. Consequently, in the interval $[0, \tau_\varepsilon]$ the function $x_N(t)$ satisfies Eqs (1.8) and the inequality $f(x) \leqslant 0$. Hence, when $0 \leqslant t \leqslant \tau$ the functions $x_*(t)$ (of (6.1)) and $x^\wedge(t)$ are identical.

Let us assume that $\mu(t) < 0$. By continuity, $\mu(t)$ is negative in some neighbourhood of τ. Then $x_n^{\cdot}(\tau_\varepsilon) > 0$ for small $\varepsilon > 0$ and a $\delta > 0$ exists such that, for $\tau_\varepsilon < t \leqslant \tau_\varepsilon + \delta$, the motion of the system occurs in the half-space $x_n > 0$. Letting ε tend to zero, we obtain the desired conclusion. The theorem is proved.

Consider a simple example. Let a point of unit mass move in the x, y plane, assuming that in the left half-plane ($x \leqslant 0$) the force applied to the point has components $0, -g$ ($g = \text{const} > 0$) but in the right half-plane ($x > 0$) the components are $0, g$. Consider motion subject to the constraint $y \geqslant 0$ and with initial data

$$x(0) = -1, \quad y(0) = 0, \quad x^{\cdot}(0) = 1, \quad y^{\cdot}(0) = 0 \tag{6.4}$$

By the classical equations of motion, the point will leave the constraint at time $t = 1$ (when $x = 0$). The function $\mu(t)$ is given by

$$\mu(t) = g, \quad t \leqslant 1; \quad \mu(t) = -g, \; t > 1$$

The first simple zero τ of the integral of $\mu(t)$ is 2. Thus, in the (0, 1, 0)-model the point must be released from the constraint at time $t = 2$.

This result may be derived directly. Replace the action of the constraint by a viscous friction force with components $0, -Ny^{\cdot}$ (in the domain $y \leqslant 0$). Then the solution with initial data (6.4) is given by the formulae

$$\begin{aligned}&x(t) = t - 1; \quad y(t) = \frac{g}{N^2}e^{-Nt} - \frac{g}{N}t + \frac{g}{N^2}, \quad t \leqslant 1\\ &y(t) = \frac{g}{N^2}(2e^N - 1)e^{-Nt} + \frac{g}{N}t - \frac{2g}{N}\left(1 + \frac{1}{2N}\right), \quad t \geqslant 1\end{aligned} \tag{6.5}$$

These formulae hold as long as $y \leqslant 0$, i.e. $t \leqslant \tau = 2 + O(1/N)$. When $t > \tau$ the point will describe a parabola in the upper half-plane. Letting N tend to infinity, we obtain the above result as to the release time of the constraint.

Let us now consider a different model of the interaction of the point with a barrier: the friction force is non-zero only when $y < 0$ and $y^{\cdot} < 0$. Then formulae (6.5) hold for $t \leqslant 1 + O(1/N)$. For large t the point will describe a parabola in the half-plane $y > 0$. In the limit as $N \to \infty$ the point will leave the constraint at time $t = 1$ (as in the classical model).

This last observation may be generalized. Let us assume that the reaction of the barrier $-\partial\Phi_N/\partial x^{\cdot}$ is non-zero only when $f < 0$ and $f^{\cdot} < 0$. At all other times it is assumed equal to zero. In other words, the barrier cannot "retain" the system. Let $\mu(0) > 0$ and let τ be the first simple zero of $\mu(t)$. It can be shown that if $x_N(t)$ is the motion of a system with initial data (1.3) subject to the aforementioned viscous friction force, then as $N \to \infty$ the function $x_N(t)$ will tend to the classical motion of the system with a unilateral constraint $f \geqslant 0$. In particular, $t = \tau$ is the release time of the constraint. Indeed, according to (6.2) and (6.3), the friction force will vanish when

$$(x_N)_n^{\cdot} = -\varepsilon\mu(t) + o(\varepsilon) \tag{6.6}$$

Since τ is the first simple zero of μ, it follows from the implicit function theorem that the first positive zero of Eq. (6.6) will be $\tau_\varepsilon = \tau + O(\varepsilon)$, and at time τ_ε the coordinate x_n and velocity $x_n^{\cdot}$ will be $O(\varepsilon)$. Letting ε tend to zero, we obtain the desired conclusion. This result may prove useful in the numerical solution of differential equations with unilateral constraints: the introduction of viscosity has a stabilizing effect in numerical methods of integration (cf. [5]).

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